INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 7, 911–917 (1987)

FAST FINITE DIFFERENCE SOLUTION FOR STEADY-STATE NAVIER–STOKES EQUATIONS USING THE BID METHOD

R. C. MITTAL AND P. K. SHARMA

Department of Mathematics, University of Roorkee, Roorkee-247667 (U.P.), India

SUMMARY

A first biharmonic boundary value problem is obtained by combining the coupled steady-state Navier–Stokes equations in their stream-function–vorticity formulation. This biharmonic boundary value problem is solved by a fast biharmonic solver developed by the authors wherein the idea of preconditioned conjugate gradient method is used. The biharmonic driver (BID) method using this solver has been found fast converging, and produces accurate results up to moderately large Reynolds numbers. Also, the mesh size does not affect the convergence rate.

KEY WORDS Finite Difference Square Driven Cavity Conjugate Gradient Method BID Method

INTRODUCTION

The Navier–Stokes equations for two-dimensional steady flow of an incompressible fluid may be written in the stream-function–vorticity formulation as

$$\nabla^2\psi = -\omega \tag{1}$$

and

$$\nabla^2\omega - R[(\psi_y\omega)_x - (\psi_x\omega)_y] = 0, \tag{2}$$

where ψ is the stream function, ω the vorticity and R the Reynolds number. Equations (1) and (2) together with given boundary conditions constitute a non-linear elliptic boundary value problem and the degree of non-linearity increases with the Reynolds number.

There are several approaches to deal with the equations (1) and (2). The first approach consists of solving the equations (1) and (2) in the coupled form. Each of the equations (1) and (2) is discretized by using a 5-point or 9-point formula to obtain a block tridiagonal form. This approach has been adopted by Burggraf,[1] Roache[2] and Gupta.[3] The same approach with some modifications has been applied by Spalding,[4] Dennis and Walsh[5] and Chein.[6] The lagging boundary conditions create a problem in this approach.

The second approach is to obtain a biharmonic boundary value problem in ψ by combining equations (1) and (2) and transferring the non-linear term to the right hand side. The biharmonic equation is discretized using a 13-point or 25-point formula, and this in turn leads to a system of algebraic equations which are in block five-diagonal form. The earlier techniques to solve a block five-diagonal form were iterative techniques, developed by Conte and Dames,[7] Fairweather *et al.*[8] and Hadjidimos.[9] These methods did not prove satisfactory for the present problem since they take a large number of iterations to converge, and do not yield a highly accurate solution.

0271-2091/87/090911-07$05.00

Received 13 September 1984
Revised 22 December 1986

Recently, some direct solvers have been devised by Bauer and Reiss,[10] Buzbee and Dorr[11] and Gupta and Manohar.[12] Roache and Ellis[13] used these solvers[10,11] for the solution of the driven cavity flow problem. They reported results up to $R = 50$. The problem for higher Reynolds numbers in these methods arises due to the increase in the number of iterations, which ultimately leads to instability. The operation count for these methods varies between $O(N^{5/2} \log N)$ and $O(N^4)$, where N^2 is the number of linear equations. The present authors have developed a biharmonic solver[14] using the preconditioned conjugate gradient method,[15] which involves $O(^{7/3})$ operations. There are numerous references to the encouraging improvements in speed and efficiency with which the conjugate-gradient method can be applied to solve algebraic equations generated by finite-difference approximations. The recent work of Khosla and Rubin[16] has brought this point out most clearly. They show that, for finite-differenced Laplace equations, the conjugate-gradient method converges to a solution an order of magnitude faster than the more commonly used point successive relaxation, successive line relaxation, alternating direction implicit, and strongly implicit procedures (see also, Reference 17). A similar improvement has been noted by Kershaw,[18] who indicates that solutions are obtained 6000 times faster than by the point Gauss–Seidel methods, 200 times faster than by ADI methods, and 30 times faster than by block successive relaxation for large problems. Results have been obtained up to $R = 400$.

DERIVATION OF THE NUMERICAL SCHEME FOR THE DRIVEN-CAVITY PROBLEM

We consider a two-dimensional square cavity, as shown in Figure 1, filled with Newtonian, viscous and incompressible fluid. The fluid is forced to move by the motion of the upper surface. When the steady state is reached, the motion is governed by the equations (1) and (2) with the following boundary conditions:

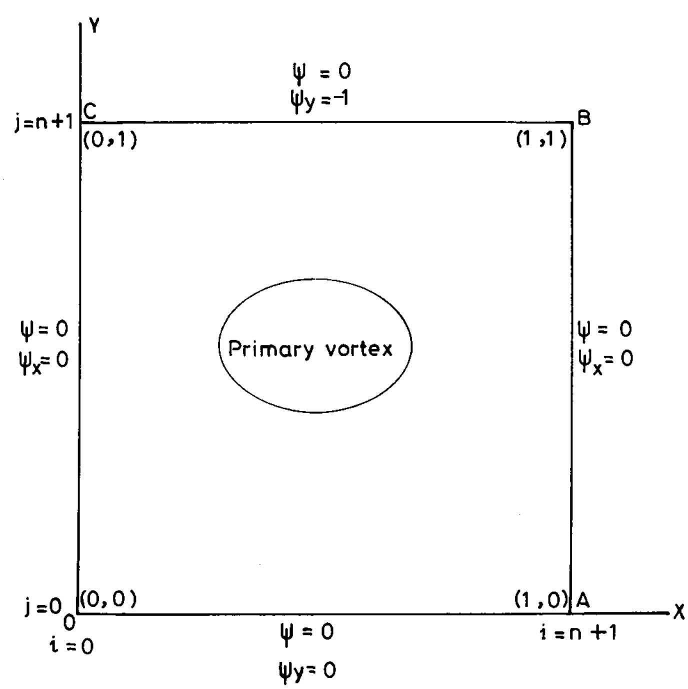

Figure 1. Driven-cavity flow problem

$$\psi = 0, \text{ along all boundaries,} \tag{3}$$

$$\psi_x = 0, \text{ along the vertical walls OC and AB,} \tag{4}$$

$$\psi_y = 0, \text{ along the bottom wall OA,} \tag{5}$$

$$\psi_y = -1, \text{ along the sliding wall BC.} \tag{6}$$

Eliminating ω between (1) and (2) and taking the non-linear term from the left-hand side to the right-hand side, the resulting equation is written as

$$\nabla^4\psi = R[(\psi_y \nabla^2\psi)_x - (\psi_x \nabla^2\psi)_y]. \tag{7}$$

The term containing the biharmonic operator is solved to compute a new iterative value $\psi^{(k)}$ by lagging the non-linear term as follows:

$$[\nabla^4\psi]^{(k)} = R[(\psi_y \nabla^2\psi)_x - (\psi_x \nabla^2\psi)_y]^{(k-1)}. \tag{8}$$

We have applied a 13-point formula for the biharmonic operator and central difference formulae for the terms of the right-hand side of equation (8). The changes in the biharmonic operator near boundaries in the interior of the domain are built into the direct solver.[14]

COMPUTATIONAL PROCEDURE

Let there be n interior points along both the directions. The following procedure is adopted for the computation.

1. Assume the initial guess values of ψ at the interior points as zeros.
2. Compute the right-hand side of equation (8) by taking the following approximation:

$$\text{R.H.S.} = h^4 R\left[\frac{\delta_{2y}\psi_{i+1,j}D^2\psi_{i+1,j} - \delta_{2y}\psi_{i-1,j}D^2\psi_{i-1,j}}{2h} - \frac{\delta_{2x}\psi_{i,j+1}D^2\psi_{i,j+1} - \delta_{2x}\psi_{i,j-1}D^2\psi_{i,j-1}}{2h}\right]^{(k-1)}, \quad i = 1,\ldots,n, \quad j = 1,\ldots,n, \tag{9}$$

where

$$D^2\psi_{i,j} = \frac{\psi_{i-1,j} + \psi_{i+1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{ij}}{h^2}$$

The points occurring outside the boundary are replaced by using the derivative boundary conditions (4)–(6).

3. The right-hand side for the mesh line $j = n$ is modified by adding the terms $2h$ for each $i = 1,\ldots, n$.
4. Solve the resulting system of linear equations by applying our algorithm[14] to obtain new values $\psi^{(k)}$.
5. Test the convergence criterion

$$\max_{ij} |\psi_{ij}^{(k)} - \psi_{ij}^{(k-1)}| < \varepsilon, \tag{10}$$

where ε is accuracy of the iterative procedure. Here, the value of ε is taken as 10^{-4}.

6. If the criterion (10) is satisfied, the computations are stopped. Otherwise, repeat the procedure (2)–(5) till the accuracy is achieved.

Table I. Values of the stream function ψ_{max} and vorticity ω_{VC} at the centre of the primary vortex, location of the primary vortex (x, y) and number of iterations with a 21×21 mesh

R	ψ_{max}	ω_{VC}	(x, y)	Iterations	Comparable results
1	0·0992	3·0018	(0·5, 0·75)	3	$\psi_{max} = 0{\cdot}0995$, $\omega_{VC} = 3{\cdot}0154$, $h = \frac{1}{20}$, CDC and CDD[3] $\psi_{max} = 0{\cdot}0993$, $\omega_{VC} = 3{\cdot}00$, $h = \frac{1}{20}$ using (1, 0) formula[23] $\psi_{max} = 0{\cdot}0995$, $\omega_{VC} = 3{\cdot}02$, $h = \frac{1}{20}$ using (2, 1) formula[23]
10	0·0991	2·9899	(0·5, 0·75)	4	$\psi_{max} = 0{\cdot}0993$, $\omega_{VC} = 3{\cdot}0010$, $h = \frac{1}{20}$, CDC[3] $\psi_{max} = 0{\cdot}0994$, $\omega_{VC} = 3{\cdot}0040$, $h = \frac{1}{20}$, CDD[3]
50	0·0993	3·2069	(0·4, 0·75)	7	$\psi_{max} = 0{\cdot}0987$, $\omega_{VC} = 3{\cdot}21$, $h = \frac{1}{20}$ using (2, 1) formula[23] $\psi_{max} = 0{\cdot}0990$, $\omega_{VC} = 3{\cdot}21$, $h = \frac{1}{20}$ using (4, 3) formula[23]
100	0·1002	3·4391	(0·35, 0·75)	10	$\psi_{max} = 0{\cdot}0971$, $\omega_{VC} = 3{\cdot}36$, $h = \frac{1}{20}$ using (2, 1) formula[23] $\psi_{max} = 0{\cdot}0965$, $\omega_{VC} = 3{\cdot}37$, $h = \frac{1}{20}$ using (3, 2) formula[23]
200	0·1037	2·9737	(0·4, 0·65)	22	
400	0·1012	2·634	(0·45, 0·6)	69	$\psi_{max} = 0{\cdot}1139$, $\omega_{VC} = 2{\cdot}2947$, $h = \frac{1}{128}$[22] $\psi_{max} = 0{\cdot}1129$, $\omega_{VC} = 2{\cdot}2810$, $h = \frac{1}{140}$[24] $\psi_{max} = 0{\cdot}1132$, $\omega_{VC} = 2{\cdot}36$, $h = \frac{1}{20}$ using a bilinear grid[21]

7. Lastly, compute the vorticity ω in the interior of the region by using the standard 5-point formula for the Laplacian operator in equation (1).
8. The vorticity ω on the boundary is computed by using formula (2-1) given in Reference 19.

NUMERICAL RESULTS AND DISCUSSION

We have obtained the numerical solution for the square driven cavity flow problem for Reynolds numbers in the range 1–400. The values of the stream function at the centre of the primary vortex ($\psi_{\max} = \max_{ij}|\psi_{ij}|$), the value of the vorticity at the vortex centre ω_{VC} and the location of the primary vortex (x, y) using a 21×21 mesh are reported in Table I along with comparable results from the literature. The streamlines and vorticity curves have been drawn in Figures 2–4 for $R = 10, 100$ and

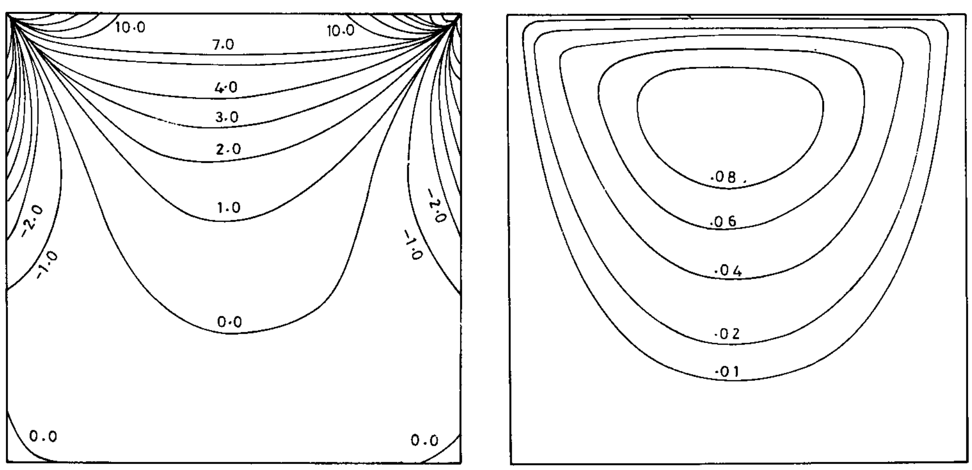

Figure 2. Equivorticity curves and streamlines for $R = 10$ with a 21×21 mesh

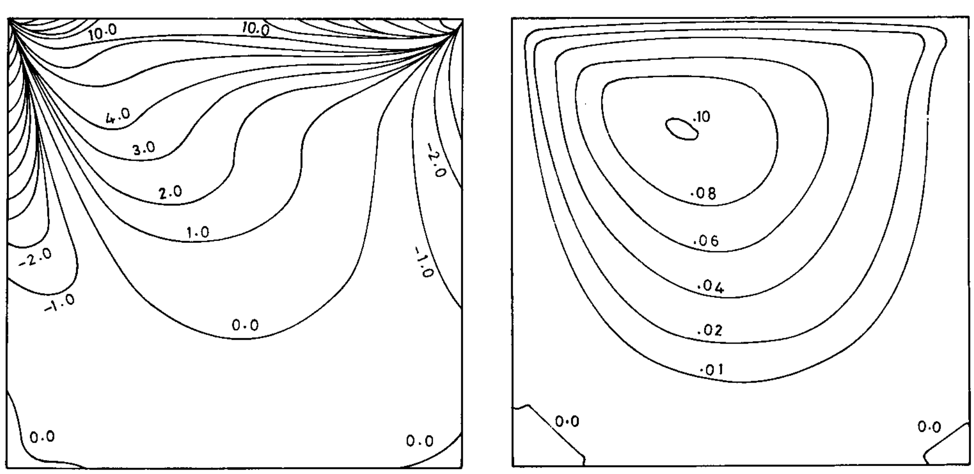

Figure 3. Equivorticity curves and streamlines for $R = 100$ with a 21×21 mesh

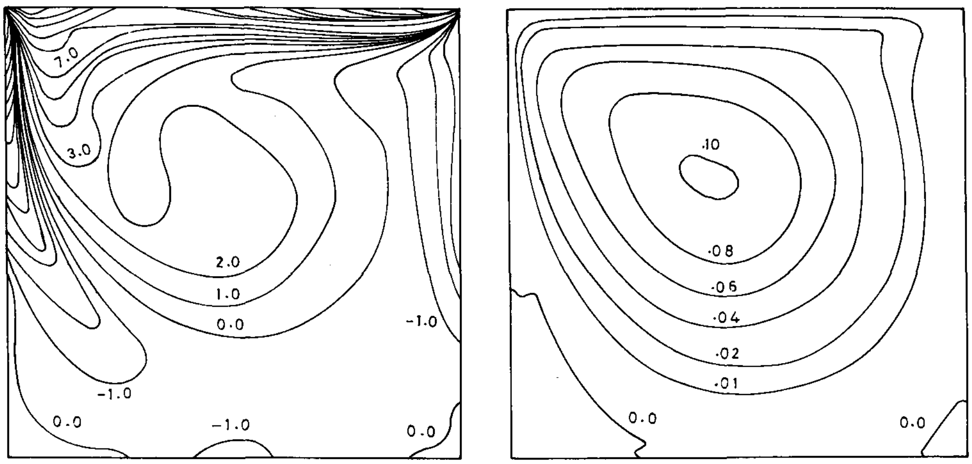

Figure 4. Equivorticity curves and streamlines for $R = 400$ with a 21×21 mesh

Table II. Comparison of convergence-rate with that of Roache and Ellis[13]

Reynolds number	Number of iterations taken by Roache and Ellis[13]	Number of iterations taken by our solver ($h = 1/20$)
10	6	2
20	8	4
50	14–18	9

400, respectively. From these Figures it is clear that there is no secondary vortex at $R = 10$, but there exist two secondary vortices at the downstream corners for $R = 100$ and 400. Also, the size of the secondary vortices increases with the increase in Reynolds number, as observed experimentally by Pan and Acrivos.[20] The equivorticity curves become more asymmetrical, and the recirculating eddies become more dominant with the increase in Reynolds number. The equivorticity curve at $R = 400$ has a secondary eddy on the bottom wall at the level $-1{\cdot}0$, as also observed by Marshall and Van Spiegel[21] and Ghia *et al.*[22] Computations have also been done for the Reynolds numbers using different mesh sizes and it has been observed that the mesh size does not affect the convergence rate.

Computations have also been made by applying the convergence criterion used by Roache and Ellis,[13] and number of iterations for the BID method using our solver[14] is compared with that of Roache and Ellis[13] in Table II. It is noted from Table II that the convergence rate of the BID method using our solver[14] is uniformly better than that of Roache and Ellis.[13]

CONCLUSIONS

The BID method presented here, using the direct solver based on the preconditioned conjugate gradient method, is more stable and faster converging than that of Roache and Ellis.[13] The results are obtained up to moderately large Reynolds numbers and these compare well with those of previous authors.[3,22–25] Also the convergence rate is independent of mesh size. The experiments of Pan and Acrivos[20] showed that the size of the downstream secondary vortex increases with R for $R < 500$. Our observations are in good agreement with this experimental behaviour.

ACKNOWLEDGEMENT

The authors gratefully acknowledge the valuable suggestions made by the referees. One of the authors (P. K. Sharma) kindly acknowledges the financial assistance given by C.S.I.R., India. All the computations have been done on the DEC-2050 system at RURCC, Roorkee.

REFERENCES

1. O. R. Burggraf, 'Analytical and numerical studies of the structure of separated flows', *J. Fluid Mech.*, **24,** 113–151 (1965).
2. P. J. Roache, 'The LAD, NOS and split NOS methods for the steady state Navier–Stokes equations', *Comput. and Fluids*, **3,** 179–195 (1975).
3. M. M. Gupta, 'A comparison of numerical solutions of convective and divergence forms of the Navier–Stokes equations for the driven-cavity problem', *J. Comput. Phys.*, **43,** 260–267 (1981).
4. D. B. Spalding, 'A novel finite difference formulation for differential expressions involving both first and second derivatives', *Int. j. numer. methods eng.*, **4,** 541–550 (1972).
5. S. C. R. Dennis and J. D. Walsh, 'Numerical solution for steady symmetric viscous flow past a parabolic cylinder in a uniform stream', *J. Fluid Mech.*, **50,** 801–814 (1971).
6. J. C. Chein, 'A general finite-difference formulation with application to Navier–Stokes equations', *Comput. and Fluids*, **5,** 15–31 (1977).
7. S. D. Conte and R. I. Dames, 'An alternating direction method for solving the biharmonic equation', *Math. Tables and Aids to Comput.*, **12,** 198–205 (1958).
8. G. Fairweather, A. R. Courlay and A. R. Mitchell, 'Some high accuracy difference schemes with a splitting operator for equations of parabolic and elliptic type', *Numerische Mathematik*, **10,** 56–66 (1967).
9. A. Hadjidimos, 'The numerical solution of a model problem biharmonic equation by using extrapolated alternating direction implicit methods', *Numerische Mathematik*, **17,** 301–317 (1971).
10. L. Bauer and E. L. Reiss, 'Block five diagonal matrices and the fast numerical solution of the biharmonic equation', *Math. Comp.*, **26,** 311–326 (1972).
11. B. L. Buzbee and F. W. Dorr, 'The direct solution of the biharmonic equation on rectangular regions and the Poisson equation on irregular regions', *SIAM J. Numer. Anal.*, **11,** 753–763 (1974).
12. M. M. Gupta and R. P. Manohar, 'Direct solution of the biharmonic equation using noncoupled approach', *J. Comp. Phys.*, **33,** 236–248 (1979).
13. P. J. Roache and M. A. Ellis, 'The BID method for the steady state Navier–Stokes equations', *Comput. and Fluids*, **3,** 305–320 (1975).
14. R. C. Mittal and P. K. Sharma, 'Fast finite difference solution of biharmonic equations on rectangles', communicated to *Computer Journal.*
15. M. R. Hestenes and E. Stiefel, 'Methods of conjugate gradients for solving linear systems', *Nat. Bur. Standards J. Research*, **49,** 409–436 (1952).
16. P. K. Khosla and S. G. Rubin, 'Navier–Stokes calculations with a coupled strongly implicit method. I. Finite difference solutions', *Comput. and Fluids*, **9,** 163–180 (1981).
17. H. L. Stone, 'Iterative solution of implicit approximations of multidimensional partial differential equations', *SIAM J. Num. Anal.*, **5,** 530–558 (1968).
18. D. S. Kershaw, 'The incomplete Cholesky-conjugate gradient method for the iterative solution of systems of linear equations', *J. Comp. Phys.*, **26,** 43–65 (1978).
19. G. de Vahl Davis and G. D. Mallinson, 'An evaluation of upwind and central difference approximations by a study of recirculating flow', *Comput. and Fluids*, **4,** 29–43 (1976).
20. F. Pan and A. Acrivos, 'Steady flows in rectangular cavities', *J. Fluid Mech.*, **28,** 643–655 (1967).
21. G. Marshall and E. Van Spiegel, 'On the numerical treatment of the Navier–Stokes equations for an incompressible fluid', *J. Eng. Math.*, **7,** 173–188 (1973).
22. U. Ghia, K. N. Ghia and C. T. Shin, 'High-Resolutions for incompressible flow using the Navier–Stokes equations and a multigrid method', *J. Comput. Phys.*, **48,** 387–411 (1982).
23. M. M. Gupta and R. P. Manohar, 'Boundary approximations and accuracy in viscous flow computations', *J. Comput. Phys.*, **31,** 265–288 (1979).
24. R. Schreiber and H. B. Keller, 'Driven cavity flows by efficient numerical techniques', *J. Comput. Phys.*, **49,** 310–333 (1983).
25. K. E. Barret and G. Demunshi, 'Finite element solutions of convective diffusion problems', *Int. j. numer. methods eng.*, **14,** 1511–1524 (1979).